2.

Calculus involving logarithmic functions

- Differentiating $y = \ln x$
- Integration to give logarithmic functions
- Miscellaneous exercise two

The *Preliminary work* section at the beginning of this book reminded us:

> Whenever we are faced with the task of finding the gradient formula, gradient function, or derivative of some 'new' function, for which we do not already have a rule, we can simply go back to the basic principle:

$$\text{Gradient at P}(x, f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Thus if $\quad y = \ln x \quad$ then $\quad \dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{\ln(x+h) - \ln(x)}{h}$.

Let us explore this limit for some values of x.

Suppose $x = 2$.

h	$\dfrac{\ln(2+h) - \ln(2)}{h}$
1	0.405 465 (6 decimal places)
0.1	0.487 902 (6 decimal places)
0.01	0.498 754 (6 decimal places)
0.001	0.499 875 (6 decimal places)
0.000 1	0.499 988 (6 decimal places)
0.000 01	0.499 999 (6 decimal places)

The table suggests that at $x = 2$, $y = \ln x$ has a gradient of 0.5, i.e. $\dfrac{1}{2}$.

Complete similar tables for $y = \ln x$ at $x = 5$, and at $x = 10$, and use your tables to suggest the gradient of $y = \ln x$ for these values of x.

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WS
Differentiating exponential and logarithmic functions

Differentiating $y = \ln x$

Did your results for the gradient of $y = \ln x$ at $x = 2$, 5 and 10 suggest that:

$$\text{If} \qquad y = \ln x$$

$$\text{then} \qquad \frac{dy}{dx} = \frac{1}{x}?$$

Rather than considering $\lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$ for other values of x we can confirm the result suggested above using the fact that $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$. (A statement that is justified at the bottom of this page.)

If $\quad y = \log_e x \quad$ then $\quad x = e^y$.

From this it follows that $\quad \frac{dx}{dy} = e^y$.

Thus $\quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

If $y = \log_e x$ then $\frac{dy}{dx} = \frac{1}{x}$.

Justification of the fact that: $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$.

We cannot simply assume this result to be true by the rules of fractions because $\frac{dy}{dx}$ is not a fraction (it is the limit of a fraction). Instead the result can be justified as follows:

Using the chain rule: $\quad \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dx} \qquad \leftarrow$ equation [1]

Now suppose that $z = x$. Differentiation gives $\frac{dz}{dx} = 1$.

Thus equation [1] becomes $\quad \frac{dx}{dy}\frac{dy}{dx} = 1 \quad$ and so $\quad \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} \quad$ as required.

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EXAMPLE 1

Differentiate **a** $3x^4 + \log_e x$ **b** $3\ln x$

Solution

a If $y = 3x^4 + \log_e x$

$$\frac{dy}{dx} = 12x^3 + \frac{1}{x}$$

b If $y = 3\ln x$

$$\frac{dy}{dx} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$$

EXAMPLE 2

Differentiate $\log_e(3x^2 + 5x)$.

Solution

If $y = \log_e(3x^2 + 5x)$

Let $u = 3x^2 + 5x$ then $y = \log_e u$.

$\frac{du}{dx} = 6x + 5$ and $\frac{dy}{du} = \frac{1}{u}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)} \\ &= \frac{1}{u} \times (6x + 5) \\ &= \frac{6x + 5}{3x^2 + 5x} \end{aligned}$$

The general statement of the above example is:

> If $y = \log_e f(x)$ then, by the chain rule, $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

EXAMPLE 3

Differentiate **a** $\log_e(3x + 5)$ **b** $\log_e(x^2 + 5)$

Solution

a If $y = \log_e(3x + 5)$

$$\frac{dy}{dx} = \frac{3}{3x + 5}$$

b If $y = \log_e(x^2 + 5)$

$$\frac{dy}{dx} = \frac{2x}{x^2 + 5}$$

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EXAMPLE 4

Differentiate $\log_e[(x+3)(x+4)]$.

Solution

$$\text{If} \quad y = \log_e[(x+3)(x+4)]$$
$$= \log_e(x^2+7x+12)$$
$$\frac{dy}{dx} = \frac{2x+7}{(x+3)(x+4)}$$

The reader should confirm that if the above question was first written as

$$\log_e(x+3) + \log_e(x+4)$$

and then differentiated the same answer would result.

Exercise 2A

Differentiate each of the following with respect to x. **For some it may be advisable to use the laws of logarithms *before* differentiating.**

1 $\log_e x$

2 $\log_e 2x$

3 $5x^2 + \log_e x$

4 $x + e^x + \log_e x$

5 $\log_e(3x+2)$

6 $\ln(2x+3)$

7 $\ln(2x-3)$

8 $\ln(x^2+1)$

9 $\ln(\cos x)$

10 $\log_e(x^2)$

11 $\log_e(\sqrt[3]{x})$

12 $\log_e(3\sqrt{x})$

13 $\log_e\left(\frac{x}{5}\right)$

14 $\log_e[x(x+3)]$

15 $\log_e[(x+4)(x-3)]$

16 $x\log_e x$

17 $(\log_e x)^3$

18 $\log_e\left(\frac{1}{x}\right)$

19 $\frac{1}{\log_e x}$

20 $e^x \log_e x$

21 $\frac{\log_e x}{x}$

22 $(1+\log_e x)^3$

23 $\log_e[x(x+5)(x+3)]$

24 $\log_e\left[\frac{x+1}{x+3}\right]$

25 $\log_e[(x^2+5)^4]$

26 $\log_e\left[\frac{x}{x^2-1}\right]$

27 $\log_e\left[\frac{(x+2)^3}{x-2}\right]$

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Find the gradient of each of the following curves at the given point on the curve.

28 $y = 7\log_e x$ at $(1, 0)$.

29 $y = x\log_e x$ at $(e^2, 2e^2)$.

30 $y = 3x^2 + \log_e x$ at $(1, 3)$.

31 $y = \dfrac{-2\log_e x}{x}$ at $(1, 0)$.

Find the exact coordinates of the point(s) on the following curves where the gradient is as stated.

32 $y = \ln x$ gradient, 0.25.

33 $y = \ln(x^2)$, gradient 4.

34 $y = \ln(6x - 5)$, gradient 0.24.

35 $y = \ln[x(x + 3)]$, gradient 0.5.

Find the equation of the tangent to the given curve at the indicated point.

36 $y = \log_e x$ at the point $(1, 0)$.

37 $y = \log_e x$ at the point $(e, 1)$.

Remembering from chapter one that $\log_a b = \dfrac{\log_c b}{\log_c a}$ differentiate each of the following.

38 $y = \log_4 x$.

39 $y = \log_6 x$.

40 If $y = 50\ln x$, use calculus to determine the approximate small change in y when x changes from 10 to 10.1.

Check your answer by evaluating $(50\ln 10.1 - 50\ln 10)$, correct to four decimal places.

41 An object moves along a straight line such that its displacement, x metres, from an origin O, at time t seconds, is given by

$$x = t + \ln t$$

Find the velocity and acceleration of the object when $t = 2$.

42 Use calculus to determine the nature and coordinates of any turning points on the graph of

$$y = x^2 - 50\ln 2x,\ x > 0.$$

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Integration to give logarithmic functions

WS Integration of reciprocals

If $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Thus $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$.

If $y = \ln f(x)$ then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$.

Thus $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$.

Any algebraic fraction for which the numerator is the derivative of the denominator will integrate to give a natural logarithmic function.

Note:

- With $\ln x$ only defined for $x > 0$ this unit will only consider:

  $\displaystyle\int \frac{1}{x}\,dx$ for $x > 0$, and $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ for $f(x) > 0$.

  Thus, for this unit:

  $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$, for $x > 0$.

  $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$, for $f(x) > 0$.

- Although it is beyond the requirements of this unit, suppose we were asked to determine $\displaystyle\int \frac{1}{x}\,dx$ for $x < 0$.

  Writing the answer as $\ln x + c$ would present a problem because we would then be faced with the logarithm of a negative number.

  However, this situation is avoidable if, for $x < 0$, we were to write $\displaystyle\int \frac{1}{x}\,dx$ as $\displaystyle\int \frac{-1}{-x}\,dx$, for which the answer is $\ln(-x) + c$.

  Thus we could say that for $x > 0$, $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$,

  and for $x < 0$, $\displaystyle\int \frac{1}{x}\,dx = \int \frac{-1}{-x}\,dx = \ln(-x) + c$.

  Combining these two statements using the absolute value gives

  $$\int \frac{1}{x}\,dx = \ln|x| + c. \qquad x \neq 0.$$

  This is mentioned here to explain why your calculator may, when asked to determine $\displaystyle\int \frac{1}{x}\,dx$, display an answer that includes the absolute value.

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In the next two examples, two methods of solution are shown.
In one method the approach is to make an intelligent first attempt at the antiderivative, differentiate it, and then use the result to adjust the first attempt appropriately. If we are attempting to antidifferentiate an expression that is of the form

$$\frac{f'(x)}{f(x)}$$

or some scalar multiple thereof, our initial attempt should be of the form

$$\ln f(x).$$

In 'method two' the given expression is first manipulated so that the task becomes that of determining

$$a\int \frac{f'(x)}{f(x)}\,dx$$

from which the answer, $a\ln f(x) + c$, follows.

The reader should be able to follow both methods but is advised to adopt whichever one they prefer.

EXAMPLE 5

Find $\int \frac{5}{2x}\,dx$ (for $x > 0$).

Solution

Intelligent guess

$$\begin{aligned} \text{Try} \quad y &= \ln 2x \\ \text{Then} \quad \frac{dy}{dx} &= \frac{2}{2x} \\ &= \frac{1}{x} \end{aligned}$$

Rearrange

$$\begin{aligned} \int \frac{5}{2x}\,dx &= \frac{5}{2}\int \frac{1}{x}\,dx \\ &= \frac{5}{2}\ln x + c. \end{aligned}$$

Our initial trial needs to be multiplied by $\frac{5}{2}$.

$$\therefore \quad \int \frac{5}{2x}\,dx = \frac{5}{2}\ln 2x + c.$$

The answers for the two methods used above may appear different but in fact they are different ways of writing the same thing:

$$\begin{aligned} \frac{5}{2}\ln 2x + c &= \frac{5}{2}[\ln 2 + \ln x] + c \\ &= \text{a constant} + \frac{5}{2}\ln x + c \\ &= \frac{5}{2}\ln x + \text{a constant}. \end{aligned}$$

EXAMPLE 6

Find $\int \frac{10x}{x^2+1}\,dx$.

Solution

Noticing that the numerator is a multiple of the derivative of the denominator we either make an intelligent guess and then adjust, or rearrange.

Intelligent guess

$$y = \ln(x^2+1).$$

Then $$\frac{dy}{dx} = \frac{2x}{x^2+1}$$

Our initial trial needs to be multiplied by 5.

$$\therefore \quad \int \frac{10x}{x^2+1}\,dx = 5\ln(x^2+1)+c.$$

Rearrange

$$\begin{aligned}\int \frac{10x}{x^2+1}\,dx &= 5\times\int \frac{2x}{x^2+1}\,dx\\ &= 5\ln(x^2+1)+c.\end{aligned}$$

With practice, the integrals can be written directly.

EXAMPLE 7

Find **a** $\int \frac{16}{2x+5}\,dx \quad (2x+5>0)$ **b** $\int \frac{15x^2}{x^3+1}\,dx \quad (x^3+1>0)$.

Solution

a $\int \frac{16}{2x+5}\,dx = 8\ln(2x+5)+c$

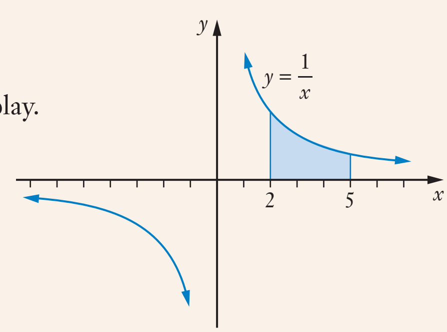
b $\int \frac{15x^2}{x^3+1}\,dx = 5\ln(x^3+1)+c$

EXAMPLE 8

Find the area between the x-axis and $y = \frac{1}{x}$ from $x = 2$ to $x = 5$.

Solution

First make a sketch or view the situation on a calculator display.



$$\begin{aligned}\text{Required area} &= \int_2^5 \frac{1}{x}\,dx\\ &= [\ln x]_2^5\\ &= \ln 5 - \ln 2\\ &= \ln 2.5\end{aligned}$$

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Exercise 2B

Find the following indefinite integrals. (Assume denominators are greater than zero.)

1 $\int \frac{5}{x}\,dx$

2 $\int \frac{4}{x}\,dx$

3 $\int \left(x + \frac{2}{x}\right) dx$

4 $\int \frac{1}{2x}\,dx$

5 $\int \frac{2x}{x^2+1}\,dx$

6 $\int \left(x^2 + \frac{5}{x}\right) dx$

7 $\int \left(4x + e^x + \frac{2}{x}\right) dx$

8 $\int \frac{2}{x+1}\,dx$

9 $\int \frac{8x}{x^2-3}\,dx$

10 $\int \frac{5}{5x-3}\,dx$

11 $\int \frac{10}{2x+1}\,dx$

12 $\int \frac{6x}{x^2+1}\,dx$

13 $\int \frac{2x+1}{x^2+x+3}\,dx$

14 $\int \frac{6x+15}{x^2+5x}\,dx$

15 $\int \frac{20x}{x^2+4}\,dx$

16 $\int \frac{\sin x}{\cos x}\,dx$

17 $\int \frac{\cos x}{\sin x}\,dx$

18 $\int \frac{\sin 2x}{\cos 2x}\,dx$

19 $\int \tan x\,dx$

20 $\int \tan 5x\,dx$

21 $\int 6\tan 2x\,dx$

22 $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$

23 $\int \frac{2+\cos 2x}{4x+\sin 2x}\,dx$

24 $\int \frac{e^x+1}{e^x+x}\,dx$

Evaluate the following definite integrals, giving **exact** answers.

25 $\int_1^3 \frac{1}{x}\,dx$

26 $\int_2^3 \frac{3}{x}\,dx$

27 $\int_1^2 \left(e^x + \frac{1}{x}\right) dx$

28 At time t seconds, $t \geq 0$, a body moving in a straight line has a displacement from an origin O of x metres and a velocity of v metres/second where

$$v = \frac{1}{t+2}.$$

If, when $t = 0$, $x = 0$, determine an expression for x in terms of t.

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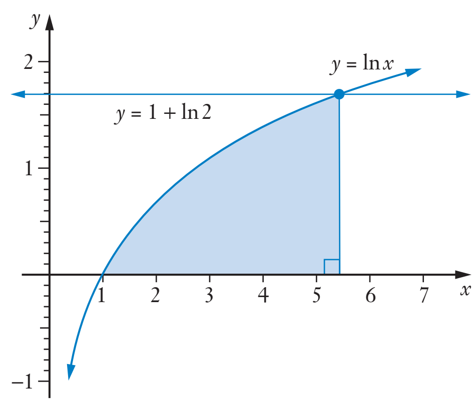
29 Find exactly, the area between $y = \dfrac{2x+1}{x}$ and the x-axis from $x = 1$ to $x = 3$.

30 Find exactly, the area enclosed by $y = \dfrac{1}{x+2} - 1$ and the axes.

31 Determine $\dfrac{dy}{dx}$ for $y = x\ln x - x$.

Hence, without the assistance of your calculator, determine the shaded area in the diagram shown on the right, giving your answer as an exact value.



32 Find the area between $y = \tan x$ and the x-axis from $x = 0$ to $x = \dfrac{\pi}{6}$, giving your answer in exact form.

33 Find the constants a and b given that for $\{x \in R: x \neq -4, x \neq -2\}$

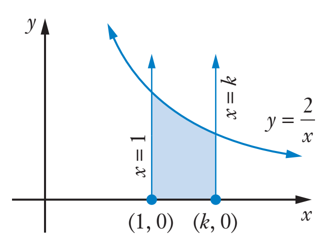
$$\frac{a}{x+4} + \frac{b}{x+2} = \frac{2(4x+13)}{(x+4)(x+2)}.$$

Hence find an expression for $\displaystyle\int \frac{2(4x+13)}{(x+4)(x+2)}\,dx \ (x > -2)$.

34 **a** Find k exactly ($k > 1$), given that the region shown shaded in the diagram has an area of 1 square unit.

b If the line $x = b$ divides the shaded region in the diagram into two regions each of area 0.5 square units, find b.

c If $(c, 0)$ is midway between $(1, 0)$ and $(k, 0)$ find the exact area between $y = \dfrac{2}{x}$ and the x–axis from $x = 1$ to $x = c$.



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Miscellaneous exercise two

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

Find $\frac{dy}{dx}$ for each of the following.

1 $y = \sin 2x$

2 $y = \cos 3x$

3 $y = e^{4x}$

4 $y = 5e^{4x}$

5 $y = \frac{2x-3}{x+1}$

6 $y = (3x-1)^4$

7 $y = 1 + 2\log_e x$

8 $y = x^2 \ln x$

9 $y = \frac{1}{x} + 3e^{2x}$

10 $y = \log_e(1 + x + x^2)$

11 Solve $2^x = 11$ giving the *exact* answer using base ten logarithms.

12 If $\log_a 5 = p$ and $\log_a 4 = q$, express each of the following in terms of p or q or both p and q:

a $\log_a 25$

b $\log_a 500$

c $\log_a 80$

d $\log_a 10$

e $\log_a(20a^3)$

f $\log_5 4$

13 Without the assistance of a calculator, determine the value of x ($x > 0$) in each of the following statements.

a $\log_x 64 = 3$

b $\log_x 64 = 2$

c $\log_x 64 = 6$

d $\log_{10} 100 = x$

e $\log 17 - \log 2 = \log x$

f $\log 17 + \log 2 = \log x$

g $\log\sqrt{2} = x\log 2$

h $3\log 2 = \log x$

14 Find an expression for p in each of the following:

a $\log_a x + \log_a y = \log_a p$

b $\log_x p = y$

c $3\log_a x - \log_a y = \log_a p$

d $2 + 0.5\log_{10} y = \log_{10} p$

15 Find the equation of the tangent to $y = \ln x$ at the point $(e^2, 2)$.

16 A pump is used to extract air from a steel container. Each minute, the pump reduces the amount of air in the container by 12% of what it was at the beginning of that minute.

Write down an expression for Q, the quantity of air in the container after t minutes pumping, in terms of Q_0, the initial quantity present.

For how long must the pump work if we require just 5% of the original amount to remain?

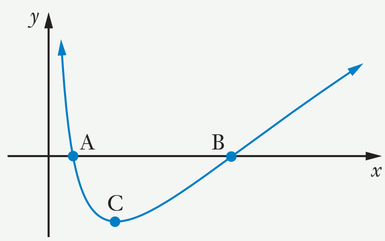
17 Given that $f'(x) = 3x^2 \ln(3x + 2)$ determine

a $f''(x)$,

b $f''(1)$.

18 The graph on the right shows part of the curve

$$y = (\log_e x)^2 - 1.$$



Answer the following without the use of a graphic calculator.

a Determine the exact coordinates of points A and B, the x-axis intercepts, and prove that there are no other places where this function cuts either axis.

b Determine the exact coordinates of point C, the local minimum, and prove that this function has no other stationary points.

c Determine whether or not $y = (\log_e x)^2 - 1$ has any points of inflection and if so determine their location.

19 Use the first principles definition:

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to prove that $\quad \dfrac{d}{dx} \sin x = \cos x \quad$ and $\quad \dfrac{d}{dx} \cos x = -\sin x.$

You may assume the following:

- $\displaystyle \lim_{h \to 0} \frac{\sin h}{h} = 1$
- $\displaystyle \lim_{h \to 0} \frac{1 - \cos h}{h} = 0$
- $\displaystyle \lim_{h \to 0} (f(h) \pm g(h)) = \lim_{h \to 0} f(h) \pm \lim_{h \to 0} g(h).$

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